

On Conjugacy of MASAs in Graph C^* -Algebras

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Abstract

For a large class of finite graphs E , we show that whenever α is a vertex-fixing quasi-free automorphism of the corresponding graph C^* -algebra $C^*(E)$ such that $\alpha(\mathcal{D}_E) \neq \mathcal{D}_E$, where \mathcal{D}_E is the canonical MASA in $C^*(E)$, then $\alpha(\mathcal{D}_E) \neq w\mathcal{D}_Ew^*$ for all unitaries $w \in C^*(E)$. That is, the two MASAs \mathcal{D}_E and $\alpha(\mathcal{D}_E)$ of $C^*(E)$ are outer but not inner conjugate. Passing to an isomorphic C^* -algebra by changing the underlying graph makes this result applicable to certain non quasi-free automorphisms as well.

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1 Introduction

Maximal abelian subalgebras (MASAs) have played very important role in the study of von Neumann algebras from the very beginning, and their theory is quite well developed by now, e.g. see [15], [14] and references therein. Theory of MASAs of C^* -algebras is somewhat less advanced, several nice attempts in this direction notwithstanding. Our particular interest lies in classification of MASAs in purely infinite simple C^* -algebras, and especially in Kirchberg algebras. In addition to its intrinsic interest, better understanding of MASAs in Kirchberg algebras could have significant consequences for the still very much open classification of automorphisms and group actions on these algebras. In this context, we would like to single out the recent work of Barlak and Li, [2], where a connection between the outstanding UCT problem for crossed products and existence of invariant Cartan subalgebras is investigated.

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It was unknown until very recently if two outer conjugate MASAs (that is, two MASAs \mathfrak{A} and \mathfrak{B} for which there exists an automorphism σ of the ambient algebra such that $\sigma(\mathfrak{A}) = \mathfrak{B}$) of a purely infinite simple C^* -algebra must necessarily be inner conjugate as well (that is, if there exists a unitary w such that $w\mathfrak{A}w^* = \mathfrak{B}$). This question was answered to the negative in [7, Theorem 3.7], with explicit counter-examples in the Cuntz algebras \mathcal{O}_n .

In the present paper, we extend the main result of [7] to the case of purely infinite simple graph C^* -algebras $C^*(E)$ corresponding to finite graphs E . Namely, we show in Theorem 3.2 below that every quasi-free automorphism of $C^*(E)$ either leaves the canonical MASA \mathcal{D}_E globally invariant or moves it to another MASA of $C^*(E)$ which is not inner conjugate to \mathcal{D}_E . Although our Theorem 3.2 is stated for quasi-free automorphisms only, it is in fact applicable to some other automorphisms as well. This is due to the fact that passing from one graph E to another F with the isomorphic algebra $C^*(F) \cong C^*(E)$ will often not preserve the property of an automorphism to be quasi-free. We discuss this phenomenon in Section 4. To make the present paper self-contained, we recall the necessary background on graph C^* -algebras and their endomorphisms in the preliminaries.

2 Preliminaries

2.1 Finite directed graphs and their C^* -algebras

Let $E = (E^0, E^1, r, s)$ be a directed graph, where E^0 and E^1 are *finite* sets of vertices and edges, respectively, and $r, s : E^1 \rightarrow E^0$ are range and source maps, respectively. A *path* μ of length $|\mu| = k \geq 1$ is a sequence $\mu = (\mu_1, \dots, \mu_k)$ of k edges μ_j such that $r(\mu_j) = s(\mu_{j+1})$ for $j = 1, \dots, k-1$. We view the vertices as paths of length 0. The set of all paths of length k is denoted E^k , and E^* denotes the collection of all finite paths (including paths of length zero). The range and source maps naturally extend from edges E^1 to paths E^k . A *sink* is a vertex v which emits no edges, i.e. $s^{-1}(v) = \emptyset$. By a *cycle* we mean a path μ of length $|\mu| \geq 1$ such that $s(\mu) = r(\mu)$. A cycle $\mu = (\mu_1, \dots, \mu_k)$ has an *exit* if there is a j such that $s(\mu_j)$ emits at least two distinct edges. Graph E is *transitive* if for any two vertices v, w there exists a path $\mu \in E^*$ from v to w of non-zero length. Thus a transitive graph does not contain any sinks or sources. Given a graph E , we will denote by $A = [A(v, w)]_{v, w \in E^0}$ its *adjacency matrix*. That is, A is a matrix with rows and columns indexed by the vertices of E , such that $A(v, w)$ is the number of edges with source v and range w .

The C^* -algebra $C^*(E)$ corresponding to a graph E is by definition, [13] and [12], the universal C^* -algebra generated by mutually orthogonal projections P_v , $v \in E^0$, and partial isometries S_e , $e \in E^1$, subject to the following two relations:

$$(GA1) \quad S_e^* S_e = P_{r(e)},$$

$$(GA2) \quad P_v = \sum_{s(e)=v} S_e S_e^* \text{ if } v \in E^0 \text{ emits at least one edge.}$$

For a path $\mu = (\mu_1, \dots, \mu_k)$ we denote by $S_\mu = S_{\mu_1} \cdots S_{\mu_k}$ the corresponding partial isometry in $C^*(E)$. We agree to write $S_v = P_v$ for a $v \in E^0$. Each S_μ is non-zero with the domain projection $P_{r(\mu)}$. Then $C^*(E)$ is the closed span of $\{S_\mu S_\nu^* : \mu, \nu \in E^*\}$. Note that $S_\mu S_\nu^*$ is non-zero if and only if $r(\mu) = r(\nu)$. In that case, $S_\mu S_\nu^*$ is a partial isometry with domain and range projections equal to $S_\nu S_\nu^*$ and $S_\mu S_\mu^*$, respectively.

The range projections $P_\mu = S_\mu S_\mu^*$ of all partial isometries S_μ mutually commute, and the abelian C^* -subalgebra of $C^*(E)$ generated by all of them is called the diagonal subalgebra and denoted \mathcal{D}_E . We set $\mathcal{D}_E^0 = \text{span}\{P_v : v \in E^0\}$ and, more generally, $\mathcal{D}_E^k = \text{span}\{P_\mu : \mu \in E^k\}$ for $k \geq 0$. C^* -algebra \mathcal{D}_E coincides with the norm closure of $\bigcup_{k=0}^\infty \mathcal{D}_E^k$. If E does not contain sinks and all cycles have exits then \mathcal{D}_E is a MASA (maximal abelian subalgebra) in $C^*(E)$ by [11, Theorem 5.2]. Throughout this paper, we make the following

standing assumption: all graphs we consider are transitive and all cycles in these graphs admit exits.

There exists a strongly continuous action γ of the circle group $U(1)$ on $C^*(E)$, called the *gauge action*, such that $\gamma_z(S_e) = zS_e$ and $\gamma_z(P_v) = P_v$ for all $e \in E^1$, $v \in E^0$ and $z \in U(1) \subseteq \mathbb{C}$. The fixed-point algebra $C^*(E)^\gamma$ for the gauge action is an AF-algebra, denoted \mathcal{F}_E and called the core AF-subalgebra of $C^*(E)$. \mathcal{F}_E is the closed span of $\{S_\mu S_\nu^* : \mu, \nu \in E^*, |\mu| = |\nu|\}$. For $k \in \mathbb{N} = \{0, 1, 2, \dots\}$ we denote by \mathcal{F}_E^k the linear span of $\{S_\mu S_\nu^* : \mu, \nu \in E^*, |\mu| = |\nu| = k\}$. C^* -algebra \mathcal{F}_E coincides with the norm closure of $\bigcup_{k=0}^\infty \mathcal{F}_E^k$.

We consider the usual *shift* on $C^*(E)$, [10], given by

$$\varphi(x) = \sum_{e \in E^1} S_e x S_e^*, \quad x \in C^*(E). \quad (1)$$

In general, for finite graphs without sinks and sources, the shift is a unital, completely positive map. However, it is an injective $*$ -homomorphism when restricted to the relative commutant $(\mathcal{D}_E^0)' \cap C^*(E)$.

We observe that for each $v \in E^0$ projection $\varphi^k(P_v)$ is minimal in the center of \mathcal{F}_E^k . The C^* -algebra $\mathcal{F}_E^k \varphi^k(P_v)$ is the linear span of partial isometries $S_\mu S_\nu^*$ with $|\mu| = |\nu| = k$ and $r(\mu) = r(\nu) = v$. It is isomorphic to the full matrix algebra of size $\sum_{w \in E^0} A^k(w, v)$. The multiplicity of $\mathcal{F}_E^k \varphi^k(P_v)$ in $\mathcal{F}_E^{k+1} \varphi^{k+1}(P_w)$ is $A(v, w)$, so the Bratteli diagram for \mathcal{F}_E is induced from the graph E , see [10], [13] or [4]. We also note that the relative commutant of \mathcal{F}_E^k in \mathcal{F}_E^{k+1} is isomorphic to $\bigoplus_{v, w \in E^0} M_{A(v, w)}(\mathbb{C})$.

For an integer $m \in \mathbb{Z}$, we denote by $C^*(E)^{(m)}$ the spectral subspace of the gauge action corresponding to m . That is,

$$C^*(E)^{(m)} := \{x \in C^*(E) \mid \gamma_z(x) = z^m x, \forall z \in U(1)\}.$$

In particular, $C^*(E)^{(0)} = C^*(E)^\gamma$. There exist faithful conditional expectations $\Phi_{\mathcal{F}} : C^*(E) \rightarrow \mathcal{F}_E$ and $\Phi_{\mathcal{D}} : C^*(E) \rightarrow \mathcal{D}_E$ such that $\Phi_{\mathcal{F}}(S_\mu S_\nu^*) = 0$ for $|\mu| \neq |\nu|$ and $\Phi_{\mathcal{D}}(S_\mu S_\nu^*) = 0$ for $\mu \neq \nu$. Combining $\Phi_{\mathcal{F}}$ with a faithful conditional expectation from \mathcal{F}_E onto \mathcal{F}_E^k , we obtain a faithful conditional expectation $\Phi_{\mathcal{F}}^k : C^*(E) \rightarrow \mathcal{F}_E^k$. Furthermore,

for each $m \in \mathbb{N}$ there is a unital, contractive and completely bounded map $\Phi^m : C^*(E) \rightarrow C^*(E)^{(m)}$ given by

$$\Phi^m(x) = \int_{z \in U(1)} z^{-m} \gamma_z(x) dx. \quad (2)$$

In particular, $\Phi^0 = \Phi_{\mathcal{F}}$. We have $\Phi^m(x) = x$ for all $x \in C^*(E)^{(m)}$. If $x \in C^*(E)$ and $\Phi^m(x) = 0$ for all $m \in \mathbb{Z}$ then $x = 0$.

2.2 Endomorphisms determined by unitaries

Cuntz's classical approach to the study of endomorphisms of \mathcal{O}_n , [9], has recently been extended to graph C^* -algebras in [7] and [1]. In this subsection, we recall a few most essential definitions and facts about such endomorphisms.

We denote by \mathcal{U}_E the collection of all those unitaries in $C^*(E)$ which commute with all vertex projections P_v , $v \in E^0$. That is

$$\mathcal{U}_E := \mathcal{U}((\mathcal{D}_E^0)' \cap C^*(E)). \quad (3)$$

If $u \in \mathcal{U}_E$ then uS_e , $e \in E^1$, are partial isometries in $C^*(E)$ which together with projections P_v , $v \in E^0$, satisfy (GA1) and (GA2). Thus, by the universality of $C^*(E)$, there exists a unital $*$ -homomorphism $\lambda_u : C^*(E) \rightarrow C^*(E)$ such that¹

$$\lambda_u(S_e) = uS_e \quad \text{and} \quad \lambda_u(P_v) = P_v, \quad \text{for } e \in E^1, v \in E^0. \quad (4)$$

The mapping $u \mapsto \lambda_u$ establishes a bijective correspondence between \mathcal{U}_E and the semi-group of those unital endomorphisms of $C^*(E)$ which fix all P_v , $v \in E^0$. As observed in [6, Proposition 1.1], if $u \in \mathcal{U}_E \cap \mathcal{F}_E$ then λ_u is automatically injective. We say λ_u is *invertible* if λ_u is an automorphism of $C^*(E)$. We denote

$$\mathfrak{B} := (\mathcal{D}_E^0)' \cap \mathcal{F}_E^1. \quad (5)$$

That is, \mathfrak{B} is the linear span of elements $S_e S_f^*$, $e, f \in E^1$, with $s(e) = s(f)$. We note that \mathfrak{B} is contained in the multiplicative domain of φ and we have $\mathcal{D}_E^1 \subseteq \mathfrak{B} \subseteq \mathcal{F}_E^1$. If $u \in \mathcal{U}(\mathfrak{B})$ then λ_u is automatically invertible with inverse λ_{u^*} and the map

$$\mathcal{U}(\mathfrak{B}) \ni u \mapsto \lambda_u \in \text{Aut}(C^*(E)) \quad (6)$$

is a group homomorphism with range inside the subgroup of *quasi-free automorphisms* of $C^*(E)$, see [16]. Note that this group is almost never trivial and it is non-commutative if graph E contains two edges $e, f \in E^1$ such that $s(e) = s(f)$ and $r(e) = r(f)$.

The shift φ globally preserves \mathcal{U}_E , \mathcal{F}_E and \mathcal{D}_E . For $k \geq 1$ we denote

$$u_k := u\varphi(u) \cdots \varphi^{k-1}(u). \quad (7)$$

For each $u \in \mathcal{U}_E$ and all $e \in E^1$ we have $S_e u = \varphi(u) S_e$, and thus

$$\lambda_u(S_\mu S_\nu^*) = u_{|\mu|} S_\mu S_\nu^* u_{|\nu|}^* \quad (8)$$

for any two paths $\mu, \nu \in E^*$.

¹The reader should be aware that in some papers (e.g. in [9]) a different convention is used, namely $\lambda_u(S_e) = u^* S_e$.

3 Quasi-free automorphisms

In this section, we extend the main result of [7], applicable to the Cuntz algebras, to a much wider class of graph C^* -algebras.

For the proof of Lemma 3.1, below, we recall from Lemma 3.2 and Remark 3.3 in [7] that if $x \in C^*(E)$, $x \geq 0$, and $x\mathcal{D}_E = \mathcal{D}_E x$ then $x \in \mathcal{D}_E$. Also, for $v, w \in E^0$ we set

$${}_v Q_w := \sum_{s(e)=v, r(e)=w} S_e S_e^*.$$

Then ${}_v Q_w$ is a minimal central projection in \mathfrak{B} , and $\mathfrak{B}_v Q_w$ is isomorphic to $M_{A(v,w)}(\mathbb{C})$. Furthermore, $P_v = \sum_{w \in E^0} {}_v Q_w$ is a central projection in \mathfrak{B} .

Lemma 3.1. *Let $u \in \mathcal{U}(\mathfrak{B})$ be such that $u\mathcal{D}_E^1 u^* \neq \mathcal{D}_E^1$, and let $x \in \mathcal{F}_E$ be arbitrary. If $x\lambda_u(\mathcal{D}_E) = \mathcal{D}_E x$ then $x = 0$.*

Proof. Suppose $x \in \mathcal{F}_E$ is such that $\|x\| = 1$ and $x\lambda_u(\mathcal{D}_E) = \mathcal{D}_E x$. From this we will derive a contradiction.

Since $u\mathcal{D}_E^1 u^* \neq \mathcal{D}_E^1$, there exists a vertex $v \in E^0$ such that $u\mathcal{D}_E^1 u^* P_v \neq \mathcal{D}_E^1 P_v$. Thus, since $u\mathcal{D}_E^1 P_v u^* = u\mathcal{D}_E^1 u^* P_v$, we can take a projection $p \in \mathcal{D}_E^1 P_v$ satisfying

$$\delta := \inf\{\|upu^* - q\| \mid q \in \mathcal{D}_E^1 P_v\} > 0.$$

Since $\Phi_{\mathcal{F}}^1(q') \in \mathcal{D}_E^1$, for all $q' \in \mathcal{D}_E$ we get

$$\|upu^* - q'\| \geq \|\Phi_{\mathcal{F}}^1(upu^* - q')\| = \|upu^* - \Phi_{\mathcal{F}}^1(q')\| \geq \delta.$$

By assumption, for each $k \in \mathbb{N}$ there is a $q_k \in \mathcal{D}_E$ such that

$$x\lambda_u(\varphi^k(p)) = q_k x. \tag{9}$$

Since $u_k \in \mathcal{F}_E^k$ and $\varphi^k(upu^*) \in \varphi^k(\mathfrak{B}) = (\mathcal{F}_E^k)' \cap \mathcal{F}_E^{k+1}$, we have

$$\lambda_u(\varphi^k(p)) = u_k \varphi^k(\lambda_u(p)) u_k^* = u_k \varphi^k(upu^*) u_k^* = \varphi^k(upu^*). \tag{10}$$

Identities (9) and (10) combined yield

$$0 = x\lambda_u(\varphi^k(p)) - q_k x = x\varphi^k(upu^*) - q_k x.$$

Since $upu^* \in \mathfrak{B}$, the sequence $\{\varphi^k(upu^*)\}_{k=1}^\infty$ is central in \mathcal{F}_E . Therefore we have

$$\lim_{k \rightarrow \infty} (\varphi^k(upu^*) - q_k) x x^* = 0.$$

It follows from the assumption on x that $xx^* \mathcal{D}_E = \mathcal{D}_E xx^*$, and thus we may conclude that $xx^* \in \mathcal{D}_E$.

Now, take an arbitrary $\epsilon > 0$. For a sufficiently large $m \in \mathbb{N}$, we have

$$\limsup_{k \rightarrow \infty} \|(\varphi^k(upu^*) - q_k) \Phi_{\mathcal{F}}^m(xx^*)\| \leq \epsilon \quad \text{and} \quad \|\Phi_{\mathcal{F}}^m(xx^*)\| \geq 1 - \epsilon.$$

Thus we can find a projection $d \in \mathcal{D}_E^m$ such that

$$\limsup_{k \rightarrow \infty} \|(\varphi^k(upu^*) - q_k)d\| \leq \frac{\epsilon}{1 - \epsilon}.$$

Since graph E is transitive, for a sufficiently large $k \in \mathbb{N}$ we can find a path $\mu \in E^k$ such that $r(\mu) = v$ and $S_\mu S_\mu^* \leq d$. But now we see that

$$3\epsilon \geq \|(\varphi^k(upu^*) - q_k)d\| \geq \|(\varphi^k(upu^*) - q_k)S_\mu S_\mu^*\| = \|upu^*P_v - S_\mu^*q_k S_\mu\| \geq \delta.$$

Since ϵ can be arbitrarily small, this is the desired contradiction. \square

Now, we are ready to prove our main result.

Theorem 3.2. *Let $u \in \mathcal{U}(\mathfrak{B})$ be such that $u\mathcal{D}_E^1 u^* \neq \mathcal{D}_E^1$. Then there is no non-zero element $x \in C^*(E)$ satisfying $x\lambda_u(\mathcal{D}_E) = \mathcal{D}_E x$. In particular, there is no unitary $w \in C^*(E)$ such that $w\mathcal{D}_E w^* = \lambda_u(\mathcal{D}_E)$.*

Proof. Let $x \in C^*(E)$ be as in the statement of the theorem. To verify that $x = 0$, it suffices to show that $\Phi^m(x) = 0$ for all $m \in \mathbb{Z}$.

We have $S_\mu^* \mathcal{D}_E S_\mu = P_{r(\mu)} \mathcal{D}_E$ for each $\mu \in E^*$. Thus $P_{r(\mu)} x \lambda_u(\mathcal{D}_E) = P_{r(\mu)} \mathcal{D}_E x = S_\mu^* \mathcal{D}_E S_\mu x$, and hence $S_\mu x \lambda_u(\mathcal{D}_E) = \mathcal{D}_E S_\mu x$. Therefore by Lemma 3.1, we get

$$\Phi_{\mathcal{F}}(S_\mu x) = 0 \quad \text{for all } \mu \in E^*.$$

Let $m \in \mathbb{N}$. For a vertex $v \in E^0$ take a path $\mu \in E^m$ with $r(\mu) = v$. Then $0 = \Phi_{\mathcal{F}}(S_\mu x) = S_\mu \Phi^{-m}(x)$. Thus $P_v \Phi^{-m}(x) = 0$, and summing over all $v \in E^0$ we see that $\Phi^{-m}(x) = 0$ for all $m \in \mathbb{N}$.

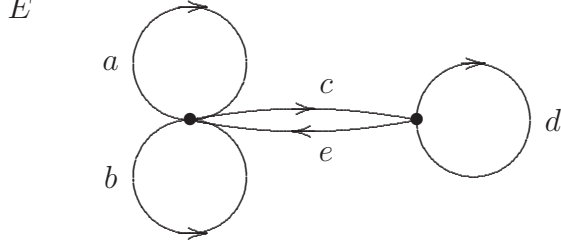
Now, taking first adjoints of both sides of the identity $x\lambda_u(\mathcal{D}_E) = \mathcal{D}_E x$ and then applying $\lambda_{u^*} = \lambda_u^{-1}$, we get $\lambda_{u^*}(x^*)\lambda_{u^*}(\mathcal{D}_E) = \mathcal{D}_E \lambda_{u^*}(x^*)$. Since $u^* \in \mathcal{U}(\mathfrak{B})$ and $u^* \mathcal{D}_E^1 u \neq \mathcal{D}_E^1$, we get $\Phi^{-m}(\lambda_{u^*}(x^*)) = 0$ for all $m \in \mathbb{N}$ applying the preceding argument. But $\Phi^{-m}(\lambda_{u^*}(x^*)) = \lambda_{u^*}(\Phi^{-m}(x^*)) = \lambda_{u^*}(\Phi^m(x))$. Thus $\Phi^m(x) = 0$ for all $m \in \mathbb{N}$, and the proof is complete. \square

Corollary 3.3. *Let $u, v \in \mathcal{U}(\mathfrak{B})$ be such that $u\mathcal{D}_E^1 u^* \neq v\mathcal{D}_E^1 v^*$. Then there is no unitary $w \in C^*(E)$ such that $w\lambda_u(\mathcal{D}_E)w^* = \lambda_v(\mathcal{D}_E)$.*

4 Changing graphs

The same graph C^* -algebra may often be presented by many different graphs, and the property of being quasi-free is usually not preserved when passing from one graph to another. This makes Theorem 3.2 applicable to a much wider class of automorphisms than quasi-free ones, and even in the case of the Cuntz algebras gives a larger class of examples of outer conjugate MASAs which are not inner conjugate by comparison with [7, Theorem 3.7]. The following Example 4.1 illustrates this phenomenon.

Example 4.1. Consider the following graph:



Then the graph algebra $C^*(E)$ is isomorphic to the Cuntz algebra $\mathcal{O}_2 = C^*(T_1, T_2)$, [8], under the identification

$$S_a = T_{11}T_1^*, \quad S_b = T_{121}T_1^*, \quad S_c = T_{122}T_2^*, \quad S_d = T_{22}T_2^*, \quad S_e = T_{21}T_1^*.$$

The inverse map is given by

$$T_1 = S_a + (S_b + S_c)(S_d + S_e)^*, \quad T_2 = S_d + S_e.$$

Note that this isomorphism carries \mathcal{D}_E onto the standard diagonal MASA \mathcal{D}_2 of \mathcal{O}_2 . Let

$$\begin{bmatrix} \xi_{aa} & \xi_{ab} \\ \xi_{ba} & \xi_{bb} \end{bmatrix}$$

be a unitary matrix with all entries non-zero complex numbers. Then

$$u = \xi_{aa}S_aS_a^* + \xi_{ab}S_aS_b^* + \xi_{ba}S_bS_a^* + \xi_{bb}S_bS_b^* + S_cS_c^* + S_dS_d^* + S_eS_e^*$$

is a unitary in \mathcal{F}_E^1 satisfying the hypothesis of Theorem 3.2. Consequently, if

$$U = \xi_{aa}T_{11}T_{11}^* + \xi_{ab}T_{11}T_{121}^* + \xi_{ba}T_{121}T_{11}^* + \xi_{bb}T_{121}T_{121}^* + T_{122}T_{122}^* + T_2T_2^*,$$

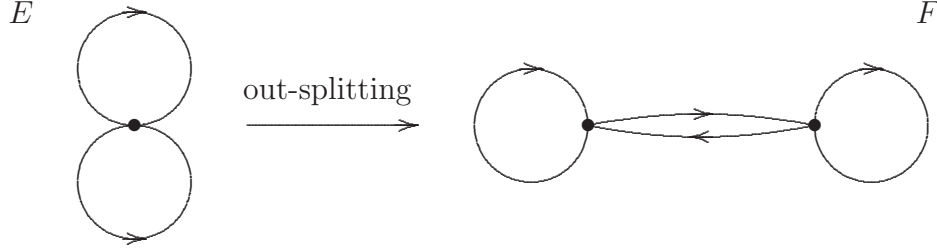
then λ_U is an automorphism of \mathcal{O}_2 (corresponding to automorphism λ_u of $C^*(E)$ via the isomorphism $\mathcal{O}_2 \cong C^*(E)$ defined above) such that there is no unitary $w \in \mathcal{O}_2$ satisfying $w\mathcal{D}_2w^* = \lambda_U(\mathcal{D}_2)$. \square

To get a broader picture, consider an *out-splitting* of a graph E , as defined in [3] (a related construction appears in [5]). Namely, for each $v \in E^0$ partition $s^{-1}(v)$ into the union of $m(v)$ non-empty, disjoint subsets $\mathcal{E}_v^1, \mathcal{E}_v^2, \dots, \mathcal{E}_v^{m(v)}$. Define a new graph F , as follows.

$$\begin{aligned} F^0 &= \{v^i \mid v \in E^0, 1 \leq i \leq m(v)\}, \\ F^1 &= \{e^j \mid e \in E^1, 1 \leq j \leq m(r(e))\}, \end{aligned}$$

with the source and the range maps defined so that for $e \in \mathcal{E}_{s(e)}^i$ we have $s(e^j) = s(e)^i$ and $r(e^j) = r(e)^j$. As shown in [3, Theorem 3.2], the C^* -algebras $C^*(E)$ and $C^*(F)$ are isomorphic by a map that carries MASA \mathcal{D}_E onto \mathcal{D}_F . However, the groups of quasi-free

automorphisms in $C^*(E)$ and $C^*(F)$ may be different. For example, in the following case



the groups $\mathcal{U}(\mathfrak{B})$ in $C^*(F)$ and $C^*(E)$ are isomorphic to $U(1) \times U(1) \times U(1) \times U(1)$ and $U(2)$, respectively.

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